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Remarks on Best Constants for Norm Inequalities among Powers of an Operator

M. K. KWONG AND A. ZETTL

Department of Mathematical Sciences, Northern Illinois University, DeKalb, Illinois 60115

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Gindler and Goldstein conjectured certain "best possible" upper bounds for the smallest constants $K(p)$ in the inequality $\| y' \|^2 \leqslant K(p) \| y \| \cdot \| y'' \|$ in the L^p spaces over the reals. Here we establish bounds on $K(p)$ which are smaller than the conjectured smallest ones, for a certain range of values of p, thus showing their conjecture to be false. In addition we construct counterexamples to another of their conjectures dealing with an operator version of this inequality in arbitrary Banach spaces. Our examples are in Hilbert space. Also we give several characterizations of these constants which, we believe, are of independent interest.

In this paper we consider the inequality

$$\| Ay \|^2 \leqslant K \| y \| \cdot \| A^2 y \| \qquad \text{for all} \quad y \in D(A^2). \tag{1.1}$$

Here A is a linear operator, in general unbounded, on a Banach space X. Clearly if (1.1) holds for some K, then there is a smallest such K. This smallest constant depends on A and X and so we denote it by $K(A, X)$.

Inequality (1) has been studied by Kallman and Rota [10] and by Kato [11] under the assumption that A is m-dissipative, i.e., for any $\alpha > 0$, $I - \alpha A$ is $1 - 1$, onto and $\|(I - \alpha A)^{-1}\| \leqslant 1$. Kallman and Rota showed that $K(A, X) \leqslant 4$ when A is m-dissipative on a Banach space X and Kato showed that $K(A, X) \leqslant 2$ when X is a Hilbert space. In both cases the constants 4 and 2 are best possible within the class of m-dissipative operators. This follows from classical results of Landau [13] and Hardy–Littlewood [8].

Gindler and Goldstein in [6, Conjecture 5.3, p. 235] make the following conjecture.

Conjecture I. If A is m-dissipative on a reflexive Banach space X, then $K(A, X) = K(A^*, X^*)$.

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An important special case of inequality (1.1) is the classical case when A is the differentiation operator: $Ay = y'$.

$$\| y' \|^2 \leqslant K \| y \| \cdot \| y'' \|. \tag{1.2}$$

Here we consider inequality (1.2) only in the Banach spaces $L^p = L^p(R)$, $1 \leqslant p \leqslant \infty$, $R = (-\infty, \infty)$. For simplicity the best constant in (1.2) is denoted by $K = K(p)$. The existence of $K(p)$ is known [8], i.e., there is a constant K, and hence a smallest such constant, such that if $y \in L^p$, y' is absolutely continuous on compact intervals and y'' is in L^p, then y' is in L^p and (1.2) holds. The known values of $K(p)$ are: $K(\infty) = 2$, Hadamard [7], $K(2) = 1$, Hardy, Littlewood and Polya [9], $K(1) = 2$, Ditzian [4]. The exact values of $K(p)$ are not known for $1 < p < \infty$, $p \neq 2$. It is known [4] that

$$K(p) \leqslant 2 = K(\infty), \qquad 1 \leqslant p \leqslant \infty.$$

In the same paper mentioned above [6] Gindler and Goldstein make a conjecture about $K(p)$.

Conjecture II. [6, p. 236].

$$K(p) \leqslant 2^{1-2/p}, \qquad 2 < p < \infty. \tag{1.3}$$

"and ⋯ we suspect equality holds."

Here we show that Conjecture I is false even in Hilbert space and, while it is possible that $2^{1-2/p}$ is an upper bound for $K(p)$, $2 < p < \infty$, the conjecture that $K(p) = 2^{1-2/p}$, $2 < p < \infty$, is also false.

The bound established below is

$$K(p) \leqslant (p-1)^{(4-p)/p}, \qquad 2 \leqslant p \leqslant 2 + 2^{1/2}. \tag{1.4}$$

This is smaller than $2^{1-2/p}$ for $3 < p \leqslant 2 + 2^{1/2}$. For instance for $p = 3.4$ we obtain $K(3.4) \leqslant (2.4)^{0.6/3.4} \doteqdot 1.167 < 2^{1-2/3.4} \doteqdot 1.330$. For $p = 3$ our bound agrees with that of Conjecture II and for $2 < p < 3$ the bound conjectured by Gindler and Goldstein is smaller than that given by (1.4). Our technique also yields bounds for larger values of p but we are unable to get a simple expression in terms of p for all $p > 2$.

The example we construct to show that Conjecture I is false is elementary. To establish inequality (1.4) we find several equivalent formulations of the problem of determining the best constant K in (1.2) in the space L^p on the whole line R. These equivalent formulations are over finite intervals for functions satisfying end point conditions. We believe these equivalent formulations are of interest in themselves.

2. Counterexamples to Conjecture I

Let A be the maximal operator generated by the differential expression y' in $L^2(0, \infty)$, i.e., $D(A) = \{y \in L^2(0, \infty) \mid y$ is absolutely continuous on all compact subintervals of $(0, \infty)$ and $y' \in L^2(0, \infty)\}$ and $Ay = y'$ for each $y \in D(A)$. Hardy and Littlewood [8] have shown that $K(A, L^2(0, \infty)) = 2$.

On the other hand the adjoint of A is the operator $B = -A$ with domain given by

$$D(-A) = \{y \in D(A) \mid y(0) = 0\}.$$

This is well known and can easily be established with a simple integration by parts. Since B is an antisymmetric operator we have $\| By \|^2 = (By, By) = -(y, B^2y) \leqslant \| y \| \cdot \| B^2y \|$ for each $y \in D(B^2)$. Hence $K(B, L^2(0, \infty)) \leqslant 1$. Thus the operator A is a counterexample to Conjecture I since it is m-dissipative on $L^2(0, \infty)$.

A general class of counterexamples to Conjecture I in Hilbert space can be constructed along similar lines. Consider a symmetric (formally self-adjoint) differential expression M given by

$$My = \sum_{j=0}^{n} (a_j y^{(j)})^{(j)}$$

where a_j are complex valued functions in $C^{2n}(R)$ and $a_n > 0$.

Let A be the maximal operator associated with M in $L^2(-\infty, \infty)$. Then A^* is the minimal operator of M. For this fact and the definition of maximal and minimal operators see [15].

Now assume that the deficiency indices of A^* are $(0, r)$ with $r \neq 0$. Such expressions M can be obtained as follows. Take any expression M with complex coefficients a_j defined on the half line $[0, \infty)$ and of order $2n$ having deficiency indices (n, s) where $s > n$. For a general construction of such expressions see Kogan and Rofe–Beketov [12]. Now extending the coefficients a_j to all of R by requiring them to be symmetric around the origin we obtain, from Kodaira's formula [15], that the new expression M on R—or equivalently its minimal operator A^*—has deficiency indices $(0, r)$ with $r > 0$.

Thus A^* is a maximal symmetric non self-adjoint operator on $L^2(R)$. Hence, by a result of Ljubic [14], $K(A, L^2(R)) = 2$ where $A = A^{**}$. On the other hand $K(A^*, L^2(R)) \leqslant 1$ since A^* is a symmetric operator.

Other differential operators A, not necessarily m-dissipative, such that $K(A^*, H) < K(A, H)$ can readily be constructed. We mention only one.

Let A be the maximal operator generated by the differential expression y'' in $L^2(0, \infty)$, i.e., $D(A) = \{y \in L^2(0, \infty)\} \, y'$ is absolutely continuous on compact subintervals of $(0, \infty)$ and y'' is in $L^2(0, \infty)\}$, and $Ay = y''$ for all $y \in D(A)$. In [3] Bradley and Everitt showed that $(8.87)^{1/2} < K(A, L^2(0, \infty)) <$

$(8.88)^{1/2}$. On the other hand it is well known that the adjoint of A is the minimal operator associated with the expression y''. Since the minimal operator $B = A^*$ is symmetric we have

$$\| By \|^2 = (By, By) = (y, B^2y) \leqslant \| y \| \| B^2y \|$$

for all y in $D(B^2)$. Hence $K(A^*, L^2(0, \infty)) \leqslant 1$.

3. Equivalent Formulations for the Determination of Best Constants

Let I denote a real interval and let $1 \leqslant p \leqslant \infty$. Define

$$M_p(I) = \{ y \in L^p(I) \mid y \not\equiv 0,\ y' \text{ is absolutely continuous on compact subintervals of } I,\ y'' \not\equiv 0 \text{ and } y'' \in L^p(I)\}.$$

For brevity we let

$$Q_p(y) = \| y' \|_p^2/(\| y \|_p \| y'' \|_p).$$

For any compact interval $I = [a, b]$ define $K_j = K_j(p, I)$, $j = 1, 2, 3, 4$, by

$$K_1 = \sup\{Q(y) \mid y \in M_p(I),\ y(a) = 0 = y(b)\} \tag{3.1}$$

$$K_2 = \sup\{Q(y) \mid y \in M_p(I),\ y(a) = 0 = y(b),\ y(t) > 0,\ a < t < b\} \tag{3.2}$$

$$K_3 = \sup\{Q(y) \mid y \in M_p(I) \mid y(a) = 0 = y'(b),\ y(t) > 0,\ a < t < b\} \tag{3.3}$$

$$K_4 = \sup\{Q(y) \mid y \in M_p(I) \mid y(a) = 0 = y'(b),\ y(t) > 0,\ y'(t) > 0,\ a < t < b\}. \tag{3.4}$$

Note that the constants K_j, $j = 1, 2, 3, 4$, do not depend on the interval I since the quotients $Q(y)$ are invariant under the change of variables $t \to ct + d$.

THEOREM 1. *For $1 \leqslant p < \infty$ and any interval $I = [a, b]$, $-\infty < a < b < \infty$ we have*

$$K(p) = K_j(p, I) = K_j(p, [0, 1]), \qquad j = 1, 2, 3, 4. \tag{3.5}$$

Proof. The proof of Theorem 1 uses two lemmas. The first one is the well known [1] fact that C_0^∞ is dense in $L^p(R)$ in the Soboljev norm.

LEMMA 1. *Let $y \in M_p(R)$, $1 \leqslant p < \infty$. For any $\delta > 0$ there exists a $g \in C_0^\infty(R)$ such that*

$$\| y - g \|_p < \delta, \qquad \| y' - g' \|_p < \delta, \qquad \| y'' - g'' \|_p < \delta. \tag{3.6}$$

LEMMA 2. *Let $h \in M_p(I)$, $1 \leqslant p < \infty$, $I = [0, 1]$. Let $J \subset I$ such that $J = \bigcup_{j=1}^n I_j$ where each I_j is an interval and the I_j's have at most end points in common. Let g be h rextricted to J and let h_j be the restriction of h to I_j, $j = 1,..., n$. If $Q(g) > \alpha > 0$, then $Q(h_j) > \alpha$ for some $j = 1,..., n$.*

Proof. It suffices to establish the lemma for the case of only two sub-intervals. Let $A = \| h_1 \|_p^p$, $B = \| h_1'' \|_p^p$, $C = \| h_2 \|_p^p$, $D = \| h_2'' \|_p^p$. $\| g^{(j)} \|_p^p = \| h_1^{(j)} \|_p^p + \| h_2^{(j)} \|_p^p$, $j = 0, 1, 2$ where each norm is taken over the appropriate interval. Suppose $Q(h_j) \leqslant \alpha$ for $j = 1, 2$. Then $\| h_1' \|_p^{2p} \leqslant \alpha^p AB$, $\| h_2' \|_p^{2p} \leqslant \alpha^p CD$ and $\| g' \|_p^{2p} = (\| h_1' \|_p^p + \| h_2' \|_p^p)^2 \leqslant \alpha^p[(AB)^{1/2} + (CD)^{1/2}]^2 = \alpha^p(AB + 2(ABCD)^{1/2} + CD)$. On the other hand, from the assumption $Q(g) > \alpha$ we get

$$\begin{aligned} \| g' \|_p^{2p} &> \alpha^p \| g \|_p^p \| g'' \|_p^p = \alpha^p(A + C)(B + D) \\ &= \alpha^p(AB + CB + AD + CD) \geqslant \alpha^p(AB + 2(ABCD)^{1/2} + CD). \end{aligned}$$

This contradiction completes the proof of Lemma 2.

Proof of Theorem 1. Clearly

$$K(p) = \sup\{Q(y) \mid y \in M_p(R), y \not\equiv 0\}.$$

Let $\epsilon > 0$. There exists an $f \in M_p(R)$ such that

$$K(p) - \epsilon < Q(f).$$

From Lemma 1 it follows that there exists a $g \in C_0^\infty(R)$ such that

$$Q(f) < Q(g) + \epsilon.$$

Hence $K_1(p, I) \geqslant Q(g) > K(p) - 2\epsilon$ where I is chosen to contain the support of g. Letting $\epsilon \to 0$ we conclude that

$$K_1(p, I) = K_1(p, [0, 1]) \geqslant K(p).$$

To show the reverse inequality, let $\epsilon > 0$, and let $g \in M_p([0, 1])$ with $g(0) = 0 = g(1)$ such that

$$K_1^p - \epsilon < (Q(g))^p.$$

Define h on $[-1, 0]$ such that h is zero in a right neighborhood of -1 and h together with g define a function on $[-1, 1]$ which has an absolutely continuous derivative and the second derivative is in $L^p[-1, 1]$. For any positive integer $n > 2$ define f_n as follows: $f_n(t) = 0$ for $t \leqslant -1$, $f_n(t) = h(t)$, $-1 \leqslant t \leqslant 0$, $f_n(t) = g(t)$, $0 \leqslant t \leqslant 1$, $f_n(t) = -g(2-t)$, $1 \leqslant t \leqslant 2$, f_n in $[2j-2, 2j]$, $j = 2,\ldots, n$, is a copy of f_n in $[0, 2]$ and f_n in $[2n, n+1]$ is a copy of h and finally $f_n(t) = 0$ for $t \geqslant 2n+1$. Geometrically, f_n is $2n$ copies of g smoothed at the ends so that f_n is a smooth function on R. From this construction we can see that

$$\| f^{(j)} \|_p^p = 2n \| g^{(j)} \|_p^p + 2A_j, \qquad j = 0, 1, 2$$

where $A_j = \| h^{(j)} \|_p^p$.

Thus

$$(Q(f_n))^p = (\| g' \|_p^p + A_1 \mid n)^2/(\| g \|_p^p + A_0/n)(\| g'' \|_p^p + A_2/n)$$
$$\to \| g' \|_p^{2p}/(\| g \|_p^p \| g'' \|_p^p) > K_1{}^p - \epsilon \qquad \text{as} \quad n \to \infty.$$

Hence

$$K^p(p) \geqslant (Q(f_n))^p > K_1{}^p - 2\epsilon.$$

Letting $\epsilon \to 0$ and taking p-th roots we have

$$K(p) \geqslant K_1 \qquad \text{and therefore} \qquad K(p) = K_1 .$$

To show that $K_1 = K_2$ all we have to show is $K_1 \leqslant K_2$. Let $\epsilon > 0$ and $h \in M_p([0, 1])$ with $h(0) = 0 = h(1)$ and

$$K_1 - \epsilon < Q(h).$$

First we consider the case when h has a finite number of zeros, say $0 \leqslant t_0 < t_1 < \cdots < t_{m+1} = 1$. Let h_j be the restriction of h to $[t_{j-1}, t_j]$. By Lemma 2, $Q(h_j) > K_1 - \epsilon$ for some j. Since the quotients $Q(y)$ are independent of the interval we can conclude that

$$K_2 \geqslant Q(| h_j |) > K_1 - \epsilon.$$

Next we consider the case when h has an infinite number of zeros in $[0, 1]$. Then the open set $\{t \in [0, 1] \mid h(t) \neq 0\} = \bigcup_{k=1}^{\infty} I_k$ where the I_k's are disjoint open (in the relative topology of $[0, 1]$) intervals in $[0, 1]$. From the additivity of the Lebesgue integral we see that

$$\lim_{m\to\infty} \sum_{k=1}^{m} \int_{I_k} | h^{(j)} |^p = \| h^{(j)} \|_p^p, \qquad j = 0, 1, 2.$$

Hence for m sufficiently large

$$\left(\sum_{k=1}^{m} \int_{I_k} | h' |^p\right)^2 \Big/ \left(\sum_{k=1}^{m} \int_{I_k} | h |^p\right)\left(\sum_{k=1}^{m} \int_{I_k} | h'' |^p\right) > K_1 - 2\epsilon.$$

Now defining h_j to be the restriction of h to I_j and applying Lemma 3 as above we conclude that $K_2 > K_1 - 2\epsilon$.

Finally if h has no zero on $(0, 1)$ and is negative we may replace h with $-h$. This completes the proof for $K_2 = K_1$.

Now to show that $K_3 = K_2$ take $I = [0, 1]$. Let $\epsilon > 0$ and choose $h \in M_p(I)$ such that $h \not\equiv 0$, $h(0) = 0 = h(1)$, $h(t) \geqslant 0$ and $Q(h) > K_2 - \epsilon$. Let $t_1 \in I$ so that $h'(t_1) = 0$. Then $0 < t_1 < 1$. Letting h_1 and h_2 denote the restrictions of h to $[0, t_1]$ and $[t_1, 1]$, respectively, and using Lemma 2 as above we obtain $K_3 \geqslant K_2$. On the other hand, for $\epsilon > 0$ choose $g \in M_p(I)$ such that $g \not\equiv 0$, $g(0) = 0 = g'(1)$, $g(t) > 0$, $0 < t < 1$ and $Q(g) > K_3 - \epsilon$. Let $f = g$ on $[0, 1]$ and define $f(t) = g(2 - t)$ for $1 \leqslant t \leqslant 2$. Then $Q(g) = Q(f)$. Finally let $h(t) = f(2t)$. Then $Q(g) = Q(f) = Q(h)$, $h(0) = 0 = h(1)$, $h(t) > 0$, $0 < t < 1$. Hence $K_2 \geqslant Q(h) = Q(g) > K_3 - \epsilon$. Consequently $K_2 \geqslant K_3$ and $K_2 = K_3$.

Clearly $K_4 \leqslant K_3$. To prove the reverse inequality, let $\epsilon > 0$ and choose $h \in M_p(I)$ for $I = [0, 1]$ satisfying $h(0) = 0 = h'(1)$, $h(t) > 0$, $0 < t < 1$ so that $Q(h) > K_4 - \epsilon$. First we consider the case when h' has a finite number of zeros in $[0, 1]$, say at the points $0 \leqslant t_1 < t_2 < \cdots < t_{m+1} = 1$. Let h_j be the restriction of h to $[t_{j-1}, t_j]$. Then, by Lemma 2, $Q(h_j) > K_4 - \epsilon$ for some j. By rescaling (again using the fact that the quotients $Q(y)$ are independent of the interval) we may assume that h_j is a function g defined on $[0, 1]$ and satisfying $g'(1) = 0$ and $g'(t) > 0$ for $0 < t < 1$. But $g(0)$ need not be zero. Since $g'(t) > 0$, $0 < t < 1$ we have that $g(t) > g(0)$ for $t \in (0, 1)$. Consider $f = g - g(0)$ and observe that $Q(f) > Q(g)$ since $\|f\|_p < \|g\|_p$ while $\|f'\| = \|g'\|$ and $\|f''\| = \|g''\|$. So $Q(f) > Q(g) = Q(h_j) > K_4 - \epsilon$ and consequently $K_3 \geqslant K_4$ in this case. The case when h' has an infinite number of zeros can be reduced to the case with a finite number of zeros in a manner similar to the proof of $K_1 = K_2$ above.

4. Upper Bounds for $K(p)$

In the determination of $K(p)$, Theorem 1 allows us to restrict ourselves to functions y defined on an arbitrary compact interval $I = [a, b]$ which are positive and have a positive derivative in (a, b). This constancy of the sign of y and y' makes the computation of the p norms more manageable.

Using the technique of integration by parts, Evans–Zettl [5] and Gindler–Goldstein [6] independently found an upper bound for $K(p)$:

$$K(p) \leqslant p - 1, \qquad 2 < p \leqslant 3.$$

With the help of Theorem 1 we can get an improvement in this bound.

Theorem 2. $K(p) \leqslant (p - 1)^{(4-p)/p}$, $2 \leqslant p \leqslant 2 + 2^{1/2}$.

Proof. By Theorem 1 it is sufficient to establish this bound for $K_4(p, I)$, $I = [0, 1]$. Let $f \in M_p(I)$ satisfy $f(0) = f'(1), f(t) > 0, f'(t) > 0, 0 < t < 1$.
In the following integrations all boundary terms vanish and all integrals are over I.

$$\int f'^p = \int f'^{p-1}f' = -(p-1)\int ff'^{p-2}f''$$
$$\leqslant (p-1)\left(\int (ff'^{p-2})^q\right)^{1/q}\left(\int |f''|^p\right)^{1/p}, \tag{4.1}$$

where $p^{-1} + q^{-1} = 1$.
Now

$$\int (ff'^{p-2})^q = \int f^r(f^{p-2}f'^2)^s = \int (f^p)^{r/p}(f^{p-2}f'^2)^s$$
$$\leqslant \left(\int f^p\right)^{r/p}\left(\int f^{p-2}f'^2\right)^s \tag{4.2}$$

where $r = 2^{-1}(2-(p-2)^2)q$, $s = p(p-2)/2(p-1)$ and noting that $r/p + s = 1$. Also

$$\int f^{p-2}f'^2 = -(p-1)^{-1}\int f^{p-1}f'' \leqslant (p-1)^{-1}\left(\int f^p\right)^{1/q}\left(\int |f''|^p\right)^{1/p}. \tag{4.3}$$

Substituting (4.3) into (4.2) and then into (4.1) we get

$$\| f' \|_p^2 \leqslant (p-1)^{(4-p)/p} \| f \|_p \| f'' \|_p .$$

The condition $2 \leqslant p \leqslant 2 + 2^{1/2}$ is imposed so that r/p and s in (4.2) satisfy $0 \leqslant r, s \leqslant 1$ (in addition to $r/p + s = 1$) so that Hölder's inequality can be used.

In [6] Gindler and Goldstein first conjectured that

$$K(p) \leqslant 2^{1-2/p} \tag{4.4}$$

and went on to say "and we suspect equality holds." Their second conjecture is false. Taking $p = 3.4$ in Theorem 2 we have

$$K(p) \leqslant (2.4)^{0.6/3.4} \doteqdot 1.167 < 2^{1-2/3.4} \doteqdot 1.330.$$

For $p = 3$ the bound given by Theorem 2 is actually the one conjectured by Gindler and Goldstein, for $3 < p < 2 + 2^{1/2}$ our bound is less than (4.4), but for $2 < p < 3$ the conjectured best bound (4.4) is less than that given in Theorem 2. We list some of our bounds and compare them with (4.4).

$$\begin{aligned} K(3.3) &\leqslant (2.2)^{0.7/3.3} \doteqdot 1.193 < 2^{1-2/3.3} \doteqdot 1.314 \\ K(3.2) &\leqslant (2.2)^{0.8/3.2} \doteqdot 1.218 < 2^{1-2/3.2} \doteqdot 1.297 \\ K(3.1) &\leqslant (2.1)^{0.9/3.1} \doteqdot 1.240 < 2^{1-2/3.1} \doteqdot 1.279 \\ K(2.5) &\leqslant (.15)^{1.5/2.5} \doteqdot 1.275 > 2^{1-2/2.5} \doteqdot 1.149. \end{aligned}$$

Our method of proof in Theorem 2 works also for $p > 2 + 2^{1/2}$ but we need to integrate by parts more often for larger values of p. Unfortunately this approach does not seem to yield a bound for $K(p)$ which can be simply expressed in terms of p. We illustrate the case $p = 4$. Let $f \in M_4([0, 1])$ with $f(0) = 0 = f'(1), f > 0, f' > 0$ on $(0, 1)$. Then

$$\begin{aligned} \int (f')^4 &= -3 \int f(f')^2 f'' \leqslant 3 \left(\int f^{4/3}(f')^{8/3}\right)^{3/4} \left(\int (f'')^4\right)^{1/4} \\ &= 3 \left((-5/7) \int f^{7/3} f'^{2/3} f''\right)^{3/4} \left(\int f''^4\right)^{1/4} \\ &\leqslant 3(5/7)^{3/4} \left(\int f^{28/9} f'^{8/9}\right)^{9/16} \left(\int f''^4\right)^{3/16} \left(\int f''^4\right)^{1/4} \end{aligned} \tag{4.5}$$

Also

$$\begin{aligned} \int f^{28/9} f'^{8/9} &= \int f^{20/9} (f^2 f'^2)^{4/9} \leqslant \left(\int f^4\right)^{5/9} \left(\int f^2 f'^2\right)^{4/9} \\ &= \left(\int f^4\right)^{5/9} \left(-3^{-1} \int f^3 f''\right)^{4/9} \\ &\leqslant 3^{-4/9} \left(\int f^4\right) \left[\left(\int f^4\right)^{3/4} \left(\int f''^4\right)^{1/4}\right]^{4/9}. \end{aligned}$$

Substituting this into (4.5) and simplifying we obtain

$$\int f'^4 \leqslant (15/7)^{3/4} \left(\int f^4\right)^{1/2} \left(\int f''^4\right)^{1/2}.$$

Hence

$$K(4) \leqslant (15/7)^{3/8} \doteqdot 1.331 < 2^{1-2/4} \doteqdot 1.414.$$

The same technique yields

$$\begin{aligned} K(5) &\leqslant 4^{94/250} (11/9)^{8/25} (19/61)^{32/125} \doteqdot 1.332 \\ K(6) &\leqslant 5^{19/108} (19/11)^{5/8} (59/91)^{25/108} \doteqdot 1.397. \end{aligned}$$

Another conjecture of Gindler and Goldstein, in the same paper, is that

$$K(p, [0, \infty)) \leqslant 2^{2/p} \qquad \text{for} \quad 1 < p < 2, \tag{4.6}$$

and that equality holds. In [2] Berdyshev showed that $K(1, [0, \infty)) = 5/2$. In a forthcoming paper we will show, by completely different methods, that $K(p, J)$ is a continuous function of p for $1 \leqslant p \leqslant \infty$ and either $J = (-\infty, \infty)$ or $J = [0, \infty)$. From this it follows that the conjecture that equality holds in (4.5) is also false.

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